Math 347: Homework 6
Due on: Oct. 31, 2018

The goal of this homework is to prove:

$$\text{Every Cauchy sequence converges.} \tag{1}$$

1. Given a sequence (a_n). A *subsequence of* (a_n) is any sequence (b_n) for which there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$b_n = a_{\varphi(n)}.$$

First we study some properties of subsequences.

   (i) Prove that if (a_n) is a convergent sequence, then any subsequence converges.

   (ii) Give examples of sequences (a_n) that don't converge but have a convergent subsequence.

   (iii) Give examples of sequences (a_n) that don't converge and don't have any convergent subsequence.

   (iv) What is a common feature of the examples in (ii) and a common feature of the examples in (iii)?

After the previous exercise one might not be surprised by the following statement

Bolzano–Weierstrass Theorem: every bounded sequence of real numbers has a convergent subsequence.

2. Suppose that a_n, b_n and c_n are three sequences such that

$$a_n \leq b_n \leq c_n$$

for all $n \geq 1$. Suppose that $\lim a_n = L$ and $\lim c_n = L$, prove that $\lim b_n = L$.

3. Let $(x_n)_{n\geq 1}$ be a sequence and suppose that there exist $M, N \in \mathbb{R}$ such that

$$N \leq x_n \leq M$$

for all $n \geq 1$. We want to inductively construct sequences a_n and c_n such that

$$N \leq a_n \leq c_n \leq M$$

for all $n \geq 1$, and $\lim a_n = \lim c_n = L$.

   (i) Find examples of such sequences (a_n) and (c_n).

   (ii) Find an example as above such that there exists a subsequence (b_n) of (x_n) such that

$$a_n \leq b_n \leq c_n$$

   for all $n \geq 1$ (Hint: for (x_n) a sequence as above and $K \in (N, M)$ either infinitely many elements of (x_n) belong to (N, K) or infinitely many elements of (x_n) belong to (K, M)[1]).

[1] Notice that this is *not* an exclusive or.

4. Use your example from Exercise 3. (ii) to prove the Bolzano-Weierstrass theorem.

5. We will want to use the Bolzano-Weierstrass Theorem to help us prove (1). For that we will need to prove that every Cauchy sequence is bounded.

6. Consider (a_n) a Cauchy sequence, by Exercise 5. we know that (a_n) is bounded and by the Bolzano-Weierstrass theorem there exists (b_n) a subsequence of (a_n) that converges to a limit L. Prove that (a_n) also converges to L (Hint: use the $\epsilon/2$-argument to chose numbers N_1 and N_2 and the definition of a subsequence to relate the b_n and a_n for sufficiently large n, i.e. n larger than both N_1 and N_2).